

# Single Jump Processes and Strict Local Martingales\*

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## Abstract

Many results in stochastic analysis and mathematical finance involve local martingales. However, specific examples of *strict* local martingales are rare and analytically often rather unhandy. We study local martingales that follow a given deterministic function up to a random time  $\gamma$  at which they jump and stay constant afterwards. The (local) martingale properties of these *single jump local martingales* are characterised in terms of conditions on the input parameters. This classification allows an easy construction of strict local martingales, uniformly integrable martingales that are not in  $H^1$ , etc. As an application, we provide a construction of a (uniformly integrable) martingale  $M$  and a bounded (deterministic) integrand  $H$  such that the stochastic integral  $H \bullet M$  is a strict local martingale.

**Keywords** Single jump; Strict local martingales; Stochastic integrals

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**JEL Classification** Y80

## 1 Introduction

Strict local martingales, i.e., local martingales which are not martingales, play an important role in mathematical finance, e.g., in the context of modelling financial bubbles [16, 4, 18, 15] or in arbitrage theory [14, 10]. Specific examples of strict local martingales are usually rather complicated, the classical example being the inverse Bessel process [5]. The aim of this paper is to study a very tractable class of processes and classify their (local) martingale properties. More precisely, we consider *single jump local martingales*, i.e., processes  $\mathcal{M}^G F = (\mathcal{M}_t^G F)_{t \in [0, \infty]}$  of the form

$$\mathcal{M}_t^G F = F(t)\mathbf{1}_{\{t < \gamma\}} + \mathcal{A}^G F(\gamma)\mathbf{1}_{\{t \geq \gamma\}}, \quad (1.1)$$

where the jump time  $\gamma$  is a  $(0, \infty)$ -valued random variable with distribution function  $G$  and  $F : [0, \infty) \rightarrow \mathbb{R}$  a function that is “locally absolutely continuous” with respect to  $G$ . In words, each path  $\mathcal{M}^G F(\omega)$  follows a deterministic function  $F$  up to some random time  $\gamma(\omega)$  and stays constant at  $\mathcal{A}^G F(\gamma(\omega))$  from time  $\gamma(\omega)$  on. The function  $\mathcal{A}^G F$  is chosen such that  $\mathcal{M}^G F$  becomes a martingale on the right-open interval  $[0, t_G)$ , where  $t_G := \sup\{t \geq 0 : G(t) < 1\} \in (0, \infty]$  denotes the *right endpoint* of the distribution function  $G$ . All local martingales studied in this article are of the form (1.1).

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The two main advantages of single jump local martingales are their flexibility and tractability. They are flexible enough to include examples of processes in well-known martingale spaces. Considered on the closed interval  $[0, t_G]$ , or equivalently on  $[0, \infty]$ ,  $\mathcal{M}^G F$  can be either of the following: not even a semimartingale; a nonintegrable local martingale; an integrable strict local martingale; a uniformly integrable martingale which does not belong to  $H^1$ ; an  $H^1$ -martingale (and of course an  $H^p$ -martingale for  $p > 1$ ). Our main result is a complete characterisation of these five cases in terms of conditions on the two input parameters  $G$  and  $F$  (cf. Figure 3.1). As for tractability, single jump local martingales are particularly suited for explicit calculations. For instance, we give a general, direct solution to the problem of finding a bounded (deterministic) integrand  $H$  and a martingale  $M$  such that the stochastic integral  $H \bullet M$  is a strict local martingale. Moreover, using only direct arguments, the authors construct in [11] counter-examples to show that neither of the no-arbitrage conditions NA and NUPBR implies the other. Because of their simple structure, these counter-examples also provide more insight into the nature of the underlying result than the more complicated counter-examples already available.

While the distribution function  $G$  of the random time  $\gamma$  is a natural input parameter, the choice of  $F$  as a second input parameter might be less clear. Another natural approach would be to start with a process  $S_t := \delta(\gamma) \mathbf{1}_{\{t \geq \gamma\}}$  for a deterministic function  $\delta : [0, \infty) \rightarrow \mathbb{R}$ . For  $\delta = 1$ , this is done in the literature on credit risk in the definition of the “hazard martingale”, see e.g. [9, Proposition 2.1]. If  $\delta$  is sufficiently integrable, the compensator (or dual predictable projection; cf. [12])  $S^p$  exists and  $M := S - S^p$  is a local martingale of the form (1.1). Yet another possibility is to start with a function  $H : (0, \infty) \rightarrow \mathbb{R}$  and to express the function  $F = (\mathcal{A}^G)^{-1} H$  in terms of  $H$  and  $G$  such that the process

$$F(t) \mathbf{1}_{\{t < \gamma\}} + H(\gamma) \mathbf{1}_{\{t \geq \gamma\}} \quad (1.2)$$

is a martingale on  $[0, t_G]$ . This is the parametrisation used in [3] and [6]; cf. the next paragraph. There are at least two reasons why we start our parametrisation with the function  $F$  instead of the jump size  $\delta$  or the function  $H$ . First, it turns out that  $F$  and  $G$  are the natural objects to decide whether  $\mathcal{M}^G F$  belongs to a certain (local) martingale space or not. For instance, if  $\mathcal{M}^G F$  is integrable, then  $\mathcal{M}^G F$  being a strict local martingale is equivalent to a nonvanishing limit  $\lim_{t \uparrow t_G} F(t)(1 - G(t))$  (cf. Lemma 3.7). If in addition  $G$  has no point mass at  $t_G$ , then  $M$  being an  $H^1$ -martingale is equivalent to  $F(\cdot)$  being  $dG$ -integrable (cf. Lemma 3.9). Second, a natural generalisation is to allow the function  $F$  to be random and to consider the corresponding process in its natural filtration (this is the subject of forthcoming work). Then the process can follow different trajectories prior to the random time  $\gamma$ , and observing its evolution corresponds to learning the conditional distribution of  $\gamma$  over time. However, if one starts with a process  $S_t = \delta \mathbf{1}_{\{t \geq \gamma\}}$  for a random variable  $\delta$ , such a learning effect is much harder to incorporate, because one would have to construct first the desired filtration and then compute the corresponding compensator. If one simply computes the compensator  $S^p$  (if it exists) in the natural filtration of  $S$ , then the local martingale  $S - S^p$  only has a single possible trajectory prior to  $\gamma$  and all information is learnt in a single instant at time  $\gamma$ .

The study of single jump processes dates back to the classical papers by Dellacherie [6] and Chou and Meyer [3]. Dellacherie [6] (see also Dellacherie and Meyer [7, Chapter IV, No. 104]) starts from the smallest filtration  $\mathbb{F}^\gamma$  with respect to which  $\gamma$  is a stopping time. Among other things, he obtains a single jump local martingale by computing the compensator of the process  $\mathbf{1}_{\{t \geq \gamma\}}$  in this filtration. He also uses single jump processes to give several counter-examples in the general theory of stochastic processes. However, his simplifying assumption that  $t_G = \infty$  immediately excludes the possibility of strict local martingales (cf. Lemma 3.1). In the same setting, Chou and Meyer [3, Proposition 1] show that any local  $\mathbb{F}^\gamma$ -martingale null at zero is a (true) martingale on  $[0, t_G]$  and of the form (1.2) with

$$F(t) = -\frac{1}{1 - G(t)} \int_{(0, t]} H(v) dG(v), \quad (1.3)$$

and that, conversely, every process of this form is a local  $\mathbb{F}^\gamma$ -martingale provided that  $H$  is “locally”  $dG$ -integrable (so that (1.3) is well defined) and  $\Delta G(t_G) = 0$ . Our Theorem 3.5 (a) corresponds

to the “converse” statement and shows that the localising sequence can be chosen to consist of stopping times with respect to the natural filtration of  $\mathcal{M}^G F$ . As this filtration is generally smaller than  $\mathbb{F}^\gamma$ , we obtain a slightly stronger statement. [3, Proposition 1] also yields that in the case of  $t_G < \infty$  and  $\Delta G(t_G) > 0$ , processes of the form (1.2) are always uniformly integrable martingales provided that  $H$  is  $dG$ -integrable. Our Theorem 3.5 (c) shows that in this case, the process is even an  $H^1$ -martingale. Single jump martingales also appear in the modelling of credit risk, see e.g. [1, 9, 13] and Remark 3.2. There the jump time models the default time of a financial asset, and single jump martingales are used to describe the *hazard function* of the default time. Note that in credit risk modelling only single jump (*true*) *martingales* are considered. To the best of our knowledge, our classification of the (local) martingale properties of single jump processes summarised in Figure 3.1 is new.

The remainder of the paper is structured as follows. Section 2 contains basic definitions and all analytic results necessary for the classification of single jump local martingales given in Section 3. Section 4 presents the counter-example in stochastic integration theory mentioned above.

## 2 Analytic preliminaries

The proof of the classification of the (local) martingale properties of single jump local martingales is split up into a purely analytic and a stochastic part. In this section, we collect all analytic preliminaries. On a first reading, the reader may wish to go only up to Definition 2.1 and then jump directly to the stochastic part in Section 3.

We always fix a distribution function  $G : \mathbb{R} \rightarrow [0, 1]$  satisfying  $G(0) = 0$  and  $G(\infty -) := \lim_{t \rightarrow \infty} G(t) = 1$ . Recall that its *right endpoint* is defined by

$$t_G := \sup\{t \geq 0 : G(t) < 1\} \in (0, \infty].$$

For notational convenience, set  $G(\infty) := 1$ . With this in mind, note that  $\Delta G(\infty) = 0$ , so that  $\Delta G(t_G) > 0$  implies  $t_G < \infty$ . Also,  $dG$  denotes the Lebesgue–Stieltjes measure on  $(0, \infty)$  induced by  $G$ , and  $L^1(dG)$  is the space of real-valued  $dG$ -integrable functions. Note that a Borel-measurable function  $\phi : (0, \infty) \rightarrow \mathbb{R}$  is  $dG$ -integrable if and only if it is  $dG$ -integrable on  $(0, t_G)$ , since  $dG$  is concentrated on  $(0, t_G]$  and a possible point mass at  $t_G$  does not affect the integrability. We call  $\phi$  *locally  $dG$ -integrable*, abbreviated by  $\phi \in L^1_{\text{loc}}(dG)$ , if  $\int_{(0, b]} |\phi(v)| dG(v) < \infty$  for each  $b \in (0, t_G)$ . Finally, we set  $\overline{G} := 1 - G$  which is often called the *survival function* of  $G$ .

### 2.1 Locally absolutely continuous functions

Classically, a Borel-measurable function  $F : [0, \infty) \rightarrow \mathbb{R}$  is called absolutely continuous on the interval  $(a, b]$  if there is a Lebesgue-integrable function  $f : (a, b] \rightarrow \mathbb{R}$  such that  $F(t) - F(a) = \int_a^t f(v) dv$  for all  $t \in (a, b]$ ; in this case,  $f$  is unique a.e. on  $(a, b]$  and is called a density of  $F$ . Replacing the Lebesgue measure by  $dG$ , we say that  $F$  is *absolutely continuous with respect to  $G$  on  $(a, b]$*  if there is a  $dG$ -integrable function  $f : (a, b] \rightarrow \mathbb{R}$  such that  $F(t) - F(a) = \int_{(a, t]} f(v) dG(v)$  for all  $t \in (a, b]$ . Unlike the Lebesgue measure,  $dG$  may have atoms, so that the precise choice of the integration domain in the previous integral is important. Our choice of a left-open and right-closed interval is natural as it forces an absolutely continuous  $F$  to be right-continuous like  $G$ . Then  $F$  itself induces a signed Lebesgue–Stieltjes measure  $dF$  on  $(a, b]$  which is absolutely continuous with respect to  $dG$  (restricted to  $(a, b]$ ) in the sense of measures, and  $f$  is a version of the Radon–Nikodým density  $\frac{dF}{dG}$  on  $(a, b]$ .

The following is a local version of this concept.

**Definition 2.1.** A Borel-measurable function  $F : [0, \infty) \rightarrow \mathbb{R}$  is called *locally absolutely continuous with respect to  $G$  on  $(0, t_G)$* , abbreviated as  $F \ll^{\text{loc}} G$ , if  $F$  is absolutely continuous with respect to  $G$  on  $(0, b]$  for all  $0 < b < t_G$ . In this case, a Borel-measurable function  $f : (0, \infty) \rightarrow \mathbb{R}$  is called a *local density of  $F$  with respect to  $G$*  if for all  $0 < b < t_G$ ,  $f$  is a version of the Radon–Nikodým density  $\frac{dF}{dG}$  on  $(0, b]$ .

The following result is an easy exercise in measure theory.

**Lemma 2.2.** *Let  $F \ll_{\text{loc}} G$ . Then  $F$  is càdlàg and of finite variation on the half-open interval  $[0, t_G)$ . Moreover, there exists a local density  $f$  of  $F$  with respect to  $G$ ; it is  $dG$ -a.e. unique on  $(0, t_G)$  and locally  $dG$ -integrable with*

$$\int_{(a,b]} f(v) dG(v) = F(b) - F(a), \quad 0 \leq a < b < t_G. \quad (2.1)$$

A local density  $f$  of  $F \ll_{\text{loc}} G$  with respect to  $G$  is only  $dG$ -a.e. unique on  $(0, t_G)$  (and not on  $[0, t_G]$ ) and may not be  $dG$ -integrable on  $(0, t_G)$ , so it may not be a classical Radon–Nikodým density. Nevertheless, we often write—in slight abuse of notation— $f = \frac{dF}{dG}$ . This is justified on the one hand by the above lemma and on the other hand by the fact that we never consider  $\frac{dF}{dG}$  outside  $(0, t_G)$ .

**Remark 2.3.** If  $F \ll_{\text{loc}} G$ , then  $F$  need not be càdlàg or of finite variation on the *right-closed* interval  $(0, t_G]$ . Indeed, define  $G : \mathbb{R} \rightarrow [0, 1]$  by  $G(t) = t\mathbf{1}_{[0,1)}(t) + \mathbf{1}_{[1,\infty)}(t)$ , i.e.,  $dG$  is a uniform distribution on  $(0, 1)$  with  $t_G = 1$ , and let  $F : [0, \infty) \rightarrow \mathbb{R}$  be given by  $F(t) = \mathbf{1}_{[0,1)}(t) \sin \frac{1}{1-t}$ . Then  $F \ll_{\text{loc}} G$  with local density

$$\frac{dF}{dG}(v) = \mathbf{1}_{(0,1)}(v) \frac{1}{(1-v)^2} \cos \frac{1}{1-v}.$$

However,  $F$  is neither càdlàg nor of finite variation on  $[0, 1]$ .

## 2.2 The function $\mathcal{A}^G F$

The first result of this section introduces and analyses the function  $\mathcal{A}^G F$  appearing in the definition of the process  $\mathcal{M}^G F$ . Its definition is motivated by the idea that  $\mathcal{M}^G F$  should be a martingale on  $[0, t_G]$  provided the function  $F$  is nice enough. We refer to the discussion after the proof of Lemma 3.1 for more details.

**Lemma 2.4.** *Let  $F \ll_{\text{loc}} G$  and define the function  $\mathcal{A}^G F : (0, \infty) \rightarrow \mathbb{R}$  by*

$$\mathcal{A}^G F(v) = \begin{cases} F(v-) - \frac{dF}{dG}(v)\overline{G}(v), & v \in (0, t_G), \\ F(t_G-) \mathbf{1}_{\{\Delta G(t_G) > 0\}}, & v \geq t_G, \text{ if } \lim_{t \uparrow t_G} F(t) \text{ exists in } \mathbb{R}, \\ 0, & v \geq t_G, \text{ if } \lim_{t \uparrow t_G} F(t) \text{ does not exist in } \mathbb{R}. \end{cases} \quad (2.2)$$

Then  $\mathcal{A}^G F \in L^1_{\text{loc}}(dG)$  and for all  $0 \leq a < b < t_G$ ,

$$\int_{(a,b]} \mathcal{A}^G F(v) dG(v) = \left[ -F(v)\overline{G}(v) \right]_a^b. \quad (2.3)$$

Thus,

$$\mathcal{A}^G F = -\frac{d(F\overline{G})}{dG} \quad dG\text{-a.e. on } (0, t_G).$$

*Proof.* Note that  $\mathcal{A}^G F$  is well defined by Lemma 2.2. To prove (2.3), fix  $0 \leq a < b < t_G$ . The function  $F(\cdot-)$  is càglàd and therefore bounded on  $(a, b]$ , the function  $\overline{G}$  is trivially bounded on  $(a, b]$ , and the function  $\frac{dF}{dG}$  is  $dG$ -integrable on  $(a, b]$  by (2.1). Thus,  $\mathcal{A}^G F \in L^1_{\text{loc}}(dG)$ . Associativity of Lebesgue–Stieltjes integrals together with an integration by parts gives the result via

$$\begin{aligned} \int_{(a,b]} \mathcal{A}^G F(v) dG(v) &= \int_{(a,b]} F(v-) dG(v) - \int_{(a,b]} \frac{dF}{dG}(v)\overline{G}(v) dG(v) \\ &= \int_{(a,b]} F(v-) dG(v) + \left[ -F(v)\overline{G}(v) \right]_a^b - \int_{(a,b]} F(v-) dG(v). \quad \square \end{aligned}$$

In general,  $\mathcal{A}^G F$  is not  $dG$ -integrable on  $(0, t_G]$ . The next result lists some equivalent conditions when this is the case and draws an important consequence.

**Lemma 2.5.** *Let  $F \ll^{\text{loc}} G$ . Then the following are equivalent:*

- (a)  $\mathcal{A}^G F \in L^1(dG)$ .
- (b)  $(\mathcal{A}^G F)^- \in L^1(dG)$  and  $\limsup_{t \uparrow \uparrow t_G} F(t) \overline{G}(t) > -\infty$ .
- (c)  $(\mathcal{A}^G F)^+ \in L^1(dG)$  and  $\liminf_{t \uparrow \uparrow t_G} F(t) \overline{G}(t) < \infty$ .

Moreover, each of the above implies that the limit  $\lim_{t \uparrow \uparrow t_G} F(t) \overline{G}(t)$  exists in  $\mathbb{R}$  and

$$\int_{(a, t_G)} \mathcal{A}^G F(v) dG(v) = F(a) \overline{G}(a) - \lim_{t \uparrow \uparrow t_G} F(t) \overline{G}(t), \quad a \in [0, t_G). \quad (2.4)$$

*Proof.* “(a)  $\Rightarrow$  (b), (c)”: If  $\mathcal{A}^G F \in L^1(dG)$ , then  $(\mathcal{A}^G F)^\pm \in L^1(dG)$ , and dominated convergence and (2.3) give

$$\begin{aligned} \int_{(0, t_G)} \mathcal{A}^G F(v) dG(v) &= \lim_{t \uparrow \uparrow t_G} \int_{(0, t]} \mathcal{A}^G F(v) dG(v) = \lim_{t \uparrow \uparrow t_G} \left[ -F(v) \overline{G}(v) \right]_0^t \\ &= F(0) - \lim_{t \uparrow \uparrow t_G} F(t) \overline{G}(t). \end{aligned}$$

This shows that  $\lim_{t \uparrow \uparrow t_G} F(t) \overline{G}(t)$  exists, and (2.4) is satisfied first for  $a = 0$  and then, by (2.3), for any  $a \in (0, t_G)$ .

“(b)  $\Rightarrow$  (a)”: Since  $(\mathcal{A}^G F)^- \in L^1(dG)$ , it suffices to show that  $\int_{(0, t_G)} \mathcal{A}^G F(v) dG(v) < \infty$ . Fatou’s lemma applied to  $(\mathcal{A}^G F)^+$ , dominated convergence for  $(\mathcal{A}^G F)^-$  and (2.3) give

$$\int_{(0, t_G)} \mathcal{A}^G F(v) dG(v) \leq \liminf_{t \uparrow \uparrow t_G} \int_{(0, t]} \mathcal{A}^G F(v) dG(v) = F(0) - \limsup_{t \uparrow \uparrow t_G} F(t) \overline{G}(t) < \infty.$$

“(c)  $\Rightarrow$  (a)”: This is analogous to the proof of “(b)  $\Rightarrow$  (a)”.  $\square$

The following result provides further characterisations of the  $dG$ -integrability of  $\mathcal{A}^G F$  in the case  $\Delta G(t_G) > 0$ . In particular, it shows that if  $\mathcal{A}^G F$  is  $dG$ -integrable, then the limit in the second line of the definition of  $\mathcal{A}^G F$  in (2.2) exists in  $\mathbb{R}$ .

**Lemma 2.6.** *Let  $F \ll^{\text{loc}} G$  and suppose that  $\Delta G(t_G) > 0$ . The following are equivalent:*

- (a)  $\mathcal{A}^G F \in L^1(dG)$ .
- (b)  $\frac{dF}{dG} \overline{G} \in L^1(dG)$ .
- (c)  $\frac{dF}{dG} \in L^1(dG)$ .
- (d)  $F$  is of finite variation on  $[0, t_G]$ .

Each of the above implies that the limit  $\lim_{t \uparrow \uparrow t_G} F(t)$  exists in  $\mathbb{R}$ .

*Proof.* The last statement follows immediately from (d), “(d)  $\Leftrightarrow$  (c)” is a standard result in analysis, and “(c)  $\Leftrightarrow$  (b)” follows immediately from the fact that the function  $\overline{G}$  is bounded above by 1 and below by  $\overline{G}(t_G -) = \Delta G(t_G) > 0$  on  $(0, t_G)$ .

For “(b)  $\Leftrightarrow$  (a)”, it suffices to show that the function  $F$  is bounded on the compact interval  $[0, t_G]$ . (Recall that  $\Delta G(t_G) > 0$  implies  $t_G < \infty$ .) Since  $F$  is càdlàg on  $[0, t_G)$ , it is enough to show that the limit  $\lim_{t \uparrow \uparrow t_G} F(t)$  exists in  $\mathbb{R}$ . Assuming (b), this follows from the equivalence “(b)  $\Leftrightarrow$  (d)” and the first part of the proof. Assuming (a), this follows via Lemma 2.5 from the fact that the limit  $\lim_{t \uparrow \uparrow t_G} F(t) \overline{G}(t)$  exists in  $\mathbb{R}$  and that  $\lim_{t \uparrow \uparrow t_G} \overline{G}(t) = \Delta G(t_G) > 0$ .  $\square$

### 2.3 Decomposition of locally absolutely continuous functions

Let  $F \ll^{\text{loc}} G$ . Define the functions  $F^\uparrow, F^\downarrow, |F| : [0, \infty) \rightarrow \mathbb{R}$  by

$$F^{\uparrow/\downarrow}(t) = \begin{cases} \int_{(0,t]} \left( \frac{dF}{dG}(v) \right)^{+/-} dG(v) & \text{if } t < t_G, \\ 0 & \text{if } t \geq t_G, \end{cases}$$

$$|F|(t) = \begin{cases} \int_{(0,t]} \left| \frac{dF}{dG}(v) \right| dG(v) & \text{if } t < t_G, \\ 0 & \text{if } t \geq t_G. \end{cases}$$

$F^\uparrow, F^\downarrow, |F|$  are well defined by Lemma 2.2, null at 0, nonnegative and increasing on  $[0, t_G)$ , and satisfy

$$F \mathbf{1}_{[0, t_G)} = F(0) \mathbf{1}_{[0, t_G)} + F^\uparrow - F^\downarrow, \quad (2.5)$$

$$|F| = F^\uparrow + F^\downarrow.$$

Restricted to  $[0, t_G)$ ,  $|F|$  is simply the total variation of  $F$  and  $F^{\uparrow/\downarrow}$  is the positive/negative variation of  $F$  shifted to null at 0.

The following result shows that if  $F \ll^{\text{loc}} G$  and  $\mathcal{A}^G F \in L^1(dG)$ , then the analogous properties hold for  $F^\uparrow, F^\downarrow, |F|$ , too.

**Lemma 2.7.** *Let  $F \ll^{\text{loc}} G$  be such that  $\mathcal{A}^G F \in L^1(dG)$ . Then  $F^\uparrow, F^\downarrow, |F| \ll^{\text{loc}} G$  and  $\mathcal{A}^G(F^\uparrow), \mathcal{A}^G(F^\downarrow), \mathcal{A}^G|F|$  are in  $L^1(dG)$ . Moreover,*

$$\mathcal{A}^G F = F(0) + \mathcal{A}^G(F^\uparrow) - \mathcal{A}^G(F^\downarrow) \quad dG\text{-a.e.}, \quad (2.6)$$

$$\mathcal{A}^G|F| = \mathcal{A}^G(F^\uparrow) + \mathcal{A}^G(F^\downarrow) \quad dG\text{-a.e.} \quad (2.7)$$

*Proof.*  $F^\uparrow, F^\downarrow, |F| \ll^{\text{loc}} G$  is clear from the definitions, and (2.6) and (2.7) are easy calculations. Among the remaining claims, we only show  $\mathcal{A}^G(F^\uparrow) \in L^1(dG)$ ;  $\mathcal{A}^G(F^\downarrow) \in L^1(dG)$  follows analogously, and then  $\mathcal{A}^G|F| \in L^1(dG)$  follows from (2.7).

We show that  $\mathcal{A}^G(F^\uparrow) \in L^1(dG)$  by using the implication “(b)  $\Rightarrow$  (a)” in Lemma 2.5. On the one hand, nonnegativity of  $F^\uparrow$  gives  $\limsup_{t \uparrow t_G} F^\uparrow(t) \overline{G}(t) \geq 0 > -\infty$ . On the other hand, (2.5) together with nonnegativity of  $F^\uparrow$  gives

$$\begin{aligned} \mathcal{A}^G(F^\uparrow) &= F^\uparrow(\cdot-) - \left( \frac{dF}{dG} \right)^+ \overline{G} \geq \left( (F(\cdot-) - F(0)) - \frac{dF}{dG} \overline{G} \right) \mathbf{1}_{\{\frac{dF}{dG} > 0\}} \\ &\geq \mathcal{A}^G F \mathbf{1}_{\{\frac{dF}{dG} > 0\}} - |F(0)| \geq -|\mathcal{A}^G F| - |F(0)| \quad dG\text{-a.e. on } (0, t_G), \end{aligned}$$

and hence,

$$\int_{(0, t_G)} (\mathcal{A}^G(F^\uparrow)(v))^- dG(v) \leq \int_{(0, t_G)} |\mathcal{A}^G F(v)| dG(v) + |F(0)| < \infty. \quad \square$$

The following lemma is in some sense the counterpart to Lemma 2.6 for the case  $\Delta G(t_G) = 0$ .

**Lemma 2.8.** *Let  $F \ll^{\text{loc}} G$  and suppose that  $\Delta G(t_G) = 0$ . The following are equivalent:*

- (a)  $F(\cdot-) \in L^1(dG)$  and  $\mathcal{A}^G F \in L^1(dG)$ .
- (b)  $F(\cdot-) \in L^1(dG)$  and  $(\mathcal{A}^G F)^- \in L^1(dG)$ .
- (c)  $F(\cdot-) \in L^1(dG)$  and  $(\mathcal{A}^G F)^+ \in L^1(dG)$ .
- (d)  $\frac{dF}{dG} \overline{G} \in L^1(dG)$ .

(e)  $|F|(\cdot-) \in L^1(dG)$ .

*Proof.* “(a)  $\Rightarrow$  (b), (c)”: This is trivial.

“(b)  $\Rightarrow$  (a)”: If  $\mathcal{A}^G F \notin L^1(dG)$ , then Lemma 2.5 implies  $\lim_{t \uparrow t_G} F(t)\overline{G}(t) = -\infty$ . Choose  $N \in \mathbb{N}$  large enough that  $\int_{(0,t_G)} |F(v-)| dG(v) < N$ , and  $t \in [0, t_G)$  such that  $|F(v)\overline{G}(v)| \geq N$  for all  $v \in [t, t_G)$ . Using  $\Delta G(t_G) = 0$ , this gives

$$\begin{aligned} \int_{(0,t_G)} |F(v-)| dG(v) &\geq \int_{(t,t_G)} \frac{|F(v-)\overline{G}(v-)|}{\overline{G}(v-)} dG(v) \\ &\geq N \int_{(t,t_G)} \frac{1}{\overline{G}(t)} dG(v) = N \frac{1-G(t)}{\overline{G}(t)} > \int_{(0,t_G)} |F(v-)| dG(v), \end{aligned}$$

which is a contradiction.

“(c)  $\Rightarrow$  (a)”: This is analogous to the proof of “(b)  $\Rightarrow$  (a)”.

“(a)  $\Rightarrow$  (d)”: This follows immediately from the definition of  $\mathcal{A}^G F$ .

“(d)  $\Leftrightarrow$  (e)”: Using the definition of  $|F|$ , Fubini’s theorem and  $\Delta G(t_G) = 0$ , we obtain

$$\begin{aligned} \int_{(0,t_G)} |F|(s-) dG(s) &= \int_{(0,t_G)} \int_{(0,s)} \left| \frac{dF}{dG}(v) \right| dG(v) dG(s) = \int_{(0,t_G)} \left| \frac{dF}{dG}(v) \right| \int_{(v,t_G)} dG(s) dG(v) \\ &= \int_{(0,t_G)} \left| \frac{dF}{dG}(v) \right| (1-G(v)) dG(v). \end{aligned}$$

This immediately establishes both directions.

“(e)  $\Rightarrow$  (a)”: On the one hand, (e) implies  $F(\cdot-) \in L^1(dG)$  as  $|F(v)| \leq |F(0)| + |F|(v)$  for  $v \in [0, t_G)$ , and on the other hand, (e) implies (d). Now the claim follows from the definition of  $\mathcal{A}^G F$ .  $\square$

**Remark 2.9.**  $F(\cdot-) \in L^1(dG)$  alone does not imply  $(\mathcal{A}^G F)^\pm \in L^1(dG)$ . Indeed, let  $G : \mathbb{R} \rightarrow [0, 1]$  be given by  $G(t) = t\mathbf{1}_{[0,1)}(t) + \mathbf{1}_{[1,\infty)}(t)$ , i.e.,  $dG$  is a uniform distribution on  $(0, 1)$  with  $t_G = 1$ , and define  $F : [0, \infty) \rightarrow \mathbb{R}$  by  $F(t) = \mathbf{1}_{[0,1)}(t) \sin \frac{1}{1-t}$ . Then an easy exercise in analysis shows

$$\begin{aligned} \int_{(0,1)} |F(v-)| dG(v) &= \int_0^1 |F(v)| dv < \infty, \\ \int_{(0,1)} (\mathcal{A}^G F(v))^\pm dG(v) &= \int_0^1 \left( \sin \frac{1}{1-v} - \frac{1}{1-v} \cos \frac{1}{1-v} \right)^\pm dv = \infty. \end{aligned}$$

### 3 Classification of single jump local martingales

Let  $(\Omega, \mathcal{F}, P)$  be a probability space and  $\gamma$  a fixed  $(0, \infty)$ -valued random variable with distribution function  $G$ . The filtration  $\mathbb{F}^\gamma = (\mathcal{F}_t^\gamma)_{t \in [0, \infty]}$  given by

$$\mathcal{F}_t^\gamma = \sigma(\{\gamma \leq s\} : s \in (0, t]) \quad (3.1)$$

is the smallest filtration with respect to which  $\gamma$  is a stopping time. For any  $F \ll_{\text{loc}} G$ , define the function  $\zeta^F : [0, \infty] \times (0, \infty) \rightarrow \mathbb{R}$  by

$$\zeta^F(t, v) = F(t)\mathbf{1}_{\{t < v\}} + \mathcal{A}^G F(v)\mathbf{1}_{\{t \geq v\}}, \quad (3.2)$$

where  $\mathcal{A}^G F : (0, \infty) \rightarrow \mathbb{R}$  is defined by

$$\mathcal{A}^G F(v) = \begin{cases} F(v-) - \frac{dF}{dG}(v)\overline{G}(v), & v \in (0, t_G), \\ F(t_G-)\mathbf{1}_{\{\Delta G(t_G) > 0\}}, & v \geq t_G, \text{ if } \lim_{t \uparrow t_G} F(t) \text{ exists in } \mathbb{R}, \\ 0, & v \geq t_G, \text{ if } \lim_{t \uparrow t_G} F(t) \text{ does not exist in } \mathbb{R}; \end{cases}$$

cf. Lemma 2.4. Note that  $\zeta^F$  is Borel-measurable and for each  $t \in [0, \infty]$ ,  $\zeta^F(t, \cdot)$  is unique up to  $dG$ -nullsets (because the local density  $\frac{dF}{dG}$  is only  $dG$ -a.e. unique on  $(0, t_G)$ ). Now define the process  $\mathcal{M}^G F = (\mathcal{M}_t^G F)_{t \in [0, \infty]}$  by

$$\mathcal{M}_t^G F = \zeta^F(t, \gamma) = F(t)\mathbf{1}_{\{t < \gamma\}} + \mathcal{A}^G F(\gamma)\mathbf{1}_{\{t \geq \gamma\}}. \quad (3.3)$$

$\mathcal{M}^G F$  is clearly  $\mathbb{F}^\gamma$ -adapted and it is easy to see that modifying  $\mathcal{A}^G F$  on a  $dG$ -nullset leads to a process that is indistinguishable from the original process  $\mathcal{M}^G F$ . Every trajectory  $\mathcal{M}_t^G F(\omega)$  is càdlàg and of finite variation on  $[0, t_G)$ , nonrandom until just before the random time  $\gamma(\omega)$ , and stays constant at  $\mathcal{A}^G F(\gamma(\omega))$  from time  $\gamma(\omega)$  on. In particular,

$$\mathcal{M}_{t_G}^G F = \mathcal{A}^G F(\gamma) \text{ } P\text{-a.s.}$$

The first line in the definition of  $\mathcal{A}^G F$  is chosen such that  $\mathcal{M}^G F$  becomes an  $\mathbb{F}^\gamma$ -martingale on the *right-open* interval  $[0, t_G)$ . This result is well known in the literature (see e.g. [6]). For the convenience of the reader, we provide a full proof here. In the following Sections 3.1–3.3, we then classify the (local) martingale properties of  $\mathcal{M}^G F$  when considered on the *closed* interval  $[0, t_G]$  (or, equivalently, on  $[0, \infty]$ ).

**Lemma 3.1.** *The process  $\mathcal{M}^G F$  is an  $\mathbb{F}^\gamma$ -martingale on  $[0, t_G)$ .*

*Proof.* For brevity, we set  $M := \mathcal{M}^G F$ . To check integrability, fix  $0 \leq t < t_G$ . Then the definition of  $M$  and Lemma 2.4 give

$$\begin{aligned} E[|M_t|] &\leq |F(t)|P[t < \gamma] + E[|\mathcal{A}^G F(\gamma)|\mathbf{1}_{\{t \geq \gamma\}}] \\ &= |F(t)|(1 - G(t)) + \int_{(0, t]} |\mathcal{A}^G F(s)| dG(s) < \infty. \end{aligned}$$

To check the martingale property for  $M$ , fix  $0 \leq s < t < t_G$ . Then  $t \geq \gamma$  on  $\{s \geq \gamma\}$  gives

$$\begin{aligned} E[M_t | \mathcal{F}_s^\gamma] &= E[M_t \mathbf{1}_{\{s < \gamma\}} + \mathcal{A}^G F(\gamma)\mathbf{1}_{\{s \geq \gamma\}} | \mathcal{F}_s^\gamma] \\ &= E[M_t \mathbf{1}_{\{s < \gamma\}} | \mathcal{F}_s^\gamma] + \mathcal{A}^G F(\gamma)\mathbf{1}_{\{s \geq \gamma\}} \text{ } P\text{-a.s.} \end{aligned}$$

It is not hard to show that  $\{\gamma > s\}$  is an atom of  $\mathcal{F}_s^\gamma$  (see e.g. [6], [3] or [7, Chapter IV, No. 104]). Using this and (2.3) gives

$$\begin{aligned} E[M_t \mathbf{1}_{\{s < \gamma\}} | \mathcal{F}_s^\gamma] &= E[M_t | s < \gamma] \mathbf{1}_{\{s < \gamma\}} \\ &= E[F(t)\mathbf{1}_{\{t < \gamma\}} + \mathcal{A}^G F(\gamma)\mathbf{1}_{\{s < \gamma \leq t\}} | s < \gamma] \mathbf{1}_{\{s < \gamma\}} \\ &= \frac{F(t)(1 - G(t)) + \int_{(s, t]} \mathcal{A}^G F(u) dG(u)}{1 - G(s)} \mathbf{1}_{\{s < \gamma\}} \\ &= \frac{F(t)\overline{G}(t) + \left[-F(u)\overline{G}(u)\right]_s^t}{\overline{G}(s)} \mathbf{1}_{\{s < \gamma\}} \\ &= \frac{F(s)\overline{G}(s)}{\overline{G}(s)} \mathbf{1}_{\{s < \gamma\}} = F(s)\mathbf{1}_{\{s < \gamma\}} \text{ } P\text{-a.s.} \end{aligned}$$

Thus, we may conclude that  $E[M_t | \mathcal{F}_s^\gamma] = F(s)\mathbf{1}_{\{s < \gamma\}} + \mathcal{A}^G F(\gamma)\mathbf{1}_{\{s \geq \gamma\}} = M_s$   $P$ -a.s.  $\square$

We are now in a position to explain the structure of the function  $\mathcal{A}^G F$ . On the one hand, if  $\Delta G(t_G) = 0$ , then  $\gamma < t_G$   $P$ -a.s. and only the first line in the definition of  $\mathcal{A}^G F$  is relevant for  $\mathcal{M}^G F$ . On  $(0, t_G)$ ,  $\mathcal{A}^G F$  is chosen such that  $\mathcal{M}^G F$  becomes a martingale on the right-open interval  $[0, t_G)$ . On the other hand, if  $\Delta G(t_G) > 0$ , then  $\gamma = t_G$  with positive probability. Assuming for the moment that  $\mathcal{M}^G F$  is a martingale on  $[0, t_G]$ , the martingale convergence theorem implies that  $\mathcal{M}_{t_G}^G F = \lim_{t \uparrow t_G} \mathcal{M}_t^G F$   $P$ -a.s. Evaluating the right-hand side on the event  $\{\gamma = t_G\}$  yields



$\mathcal{M}_{t_G}^G F = \lim_{t \uparrow t_G} F(t) = F(t_G-)$  on  $\{\gamma = t_G\}$ . This motivates the second line in the definition of  $\mathcal{A}^G F$ . The last line is only relevant when  $\Delta G(t_G) > 0$  and the left limit  $F(t_G-)$  does not exist in  $\mathbb{R}$ . But then  $F$  must be of infinite variation on  $[0, t_G]$  and

$$P[\mathcal{M}_t^G F = F(t), t \in [0, t_G]] \geq P[\gamma = t_G] = \Delta G(t_G) > 0,$$

so that  $\mathcal{M}^G F$  fails to be a semimartingale on  $[0, t_G]$  by Lemma B.6. Note that this is independent of the particular choice  $\mathcal{A}^G F(t_G) := 0$ .

**Remark 3.2.** Processes of the form  $\mathcal{M}^G F$  for particular choices of  $F$  play a special role in the modelling of credit risk, see e.g. [1, 13, 9]. We give two examples. We assume—as is usually done in the literature on credit risk—that  $t_G = \infty$ . First, for  $F := \frac{1}{1-G} = \frac{1}{\bar{G}}$ , we have  $\mathcal{A}^G F = -\frac{d(F\bar{G})}{dG} = -\frac{d1}{dG} = 0$  and

$$\mathcal{M}_t^G F = \frac{1}{1-G(t)} \mathbf{1}_{\{t < \gamma\}} + 0 \cdot \mathbf{1}_{\{t \geq \gamma\}} = \frac{1 - \mathbf{1}_{\{t \geq \gamma\}}}{1 - G(t)}, \quad t \in [0, \infty).$$

This process is called  $\hat{M}$  in [13, Corollary 5.1]. Second, for

$$F(t) := - \int_{(0,t]} \frac{dG(v)}{1 - G(v-)} = - \int_{(0,t]} \frac{dG(v)}{\bar{G}(v-)}$$

( $= \log \bar{G}(t)$  if  $G$  is continuous), we have

$$\begin{aligned} \mathcal{A}^G F(v) &= F(v-) - \frac{dF}{dG}(v) \bar{G}(v) = F(v) - \frac{dF}{dG}(v) \bar{G}(v-) \\ &= F(v) + \frac{1}{\bar{G}(v-)} \bar{G}(v-) = F(v) + 1 \quad dG\text{-a.e.} \end{aligned}$$

and

$$\mathcal{M}_t^G F = F(t) \mathbf{1}_{\{t < \gamma\}} + (F(\gamma) + 1) \mathbf{1}_{\{t \geq \gamma\}} = \mathbf{1}_{\{t \geq \gamma\}} - \int_{(0, t \wedge \gamma]} \frac{dG(v)}{1 - G(v-)}, \quad t \in [0, \infty).$$

This process is called  $M$  in [13, Proposition 5.2].

It is also often assumed that  $G$  is absolutely continuous with respect to Lebesgue measure, i.e.,  $dG(t) = G'(t) dt$  for a nonnegative Borel-measurable function  $G'$ . In this case, the quantity  $\kappa^G(t) := \frac{G'(t)}{\bar{G}(t)}$  is the conditional probability density of the default time, given that default has not

occurred up to time  $t$ , and is often called *hazard rate* or *default intensity*. Clearly, any  $F \ll^{\text{loc}} G$  is also locally absolutely continuous with respect to Lebesgue measure, i.e., there is a Borel-measurable function  $F'$  such that  $dF(t) = F'(t) dt$ . Now,  $\mathcal{M}^G F$  has the following representation in terms of  $F$ ,  $F'$  and the hazard rate of  $G$ :

$$\begin{aligned} \mathcal{M}_t^G F &= F(t) \mathbf{1}_{\{t < \gamma\}} + \mathcal{A}^G F(\gamma) \mathbf{1}_{\{t \geq \gamma\}} = F(t) \mathbf{1}_{\{t < \gamma\}} + \left( F(\gamma-) - \frac{dF}{dG}(\gamma) \bar{G}(\gamma) \right) \mathbf{1}_{\{t \geq \gamma\}}, \\ &= F(t) \mathbf{1}_{\{t < \gamma\}} + \left( F(\gamma) - \frac{F'(\gamma)}{\kappa^G(\gamma)} \right) \mathbf{1}_{\{t \geq \gamma\}}, \end{aligned}$$

or alternatively,

$$\mathcal{M}_t^G F = F(t \wedge \gamma) - \frac{F'(\gamma)}{\kappa^G(\gamma)} \mathbf{1}_{\{t \geq \gamma\}}.$$

For the rest of this section (except for Section 3.4), we fix  $F \ll^{\text{loc}} G$  and set  $M := \mathcal{M}^G F$  for brevity.

The raw filtration generated by  $M$ , denoted by  $\mathbb{F}^M = (\mathcal{F}_t^M)_{t \in [0, \infty]}$ , is the smallest filtration such that  $M$  is  $\mathbb{F}^M$ -adapted. As  $M$  is  $\mathbb{F}^\gamma$ -adapted,  $\mathbb{F}^M$  is a subfiltration of  $\mathbb{F}^\gamma$ .

**Remark 3.3.** (a) While in the filtration  $\mathbb{F}^\gamma$ , the value of  $\gamma(\omega)$  is known at time  $\gamma(\omega)$ , this may not be true for the filtration  $\mathbb{F}^M$ . In  $\mathbb{F}^M$ , we can only tell the value of  $\gamma(\omega)$  at time  $\gamma(\omega)$  if we observe a jump of  $M(\omega)$  of a certain size at time  $\gamma(\omega)$ . However, if  $\gamma(\omega) < t_G$  and  $\frac{dF}{dG}(\gamma(\omega)) = 0$ , then  $\mathcal{A}^G F(\gamma(\omega)) = F(\gamma(\omega)-)$  and  $M(\omega)$  has no jump at time  $\gamma(\omega)$  (“ $\gamma$  occurred, but we did not see it in the path of  $M$ ”). A trivial example is given by  $F \equiv 0$ . Then  $\mathcal{A}^G F \equiv 0$ ,  $M \equiv 0$ , and  $\mathbb{F}^M$  contains no information about  $\gamma$  at all.

(b) The filtrations  $\mathbb{F}^\gamma$  and  $\mathbb{F}^M$  need not be  $P$ -complete and  $\mathbb{F}^M$  need not be right-continuous in general ( $\mathbb{F}^\gamma$  is in fact right-continuous, see e.g. [12, Lemma II.3.24] or [7, Chapter IV, No. 104]). However, most of the results of martingale theory can be proved without these *usual conditions*. In particular, the martingale convergence theorem and the convergence result for stochastic integrals stated in Lemma B.6 do not rely on them.

By the law of iterated expectations, if  $M$  is an  $\mathbb{F}^\gamma$ -martingale, then it is also an  $\mathbb{F}^M$ -martingale. However, if  $M$  is a local  $\mathbb{F}^\gamma$ -martingale, then  $M$  need not be a local  $\mathbb{F}^M$ -martingale. The reason is that the  $\mathbb{F}^\gamma$ -stopping times in the localising sequence need not be  $\mathbb{F}^M$ -stopping times. To obtain stronger statements, we distinguish two filtrations in the definition of a local martingale. In particular,  $M$  is called an  $\mathbb{F}^M$ -local  $\mathbb{F}^\gamma$ -martingale if it is a local  $\mathbb{F}^\gamma$ -martingale that admits a localising sequence consisting only of  $\mathbb{F}^M$ -stopping times. We refer to Appendix B for the details and related (partly nonstandard) terminology for (semi-)martingales.

### 3.1 Local martingale property on $[0, t_G]$

The following preparatory lemma gives conditions for the integrability of  $M$  on  $[0, t_G]$ .

**Lemma 3.4.** *The following are equivalent:*

- (a) *The process  $M$  is integrable on  $[0, t_G]$ .*
- (b) *The random variable  $M_{t_G}$  is integrable.*
- (c)  *$\mathcal{A}^G F \in L^1(dG)$ .*

*Proof.* “(a)  $\Rightarrow$  (b)” is trivial, and “(b)  $\Rightarrow$  (a)” holds because  $M_t$  is integrable for  $t \in [0, t_G]$  by Lemma 3.1. “(b)  $\Leftrightarrow$  (c)” follows from  $M_{t_G} = \mathcal{A}^G F(\gamma)$   $P$ -a.s. and the fact that  $\gamma$  has distribution function  $G$  under  $P$ .  $\square$

**Theorem 3.5.**

- (a) *If  $\Delta G(t_G) = 0$ , then  $M$  is an  $\mathbb{F}^M$ -local  $\mathbb{F}^\gamma$ -martingale on  $[0, t_G]$ .*
- (b) *If  $\Delta G(t_G) > 0$  and  $M_{t_G}$  is **not** integrable, then  $M$  **fails** to be a semimartingale on  $[0, t_G]$ .*
- (c) *If  $\Delta G(t_G) > 0$  and  $M_{t_G}$  is integrable, then  $M$  is an  $H^1$ - $\mathbb{F}^\gamma$ -martingale on  $[0, t_G]$ .*

*Proof.* (a) We distinguish two cases for  $F$ . If there exists  $t^* \in [0, t_G]$  such that  $F(t) = F(t^*)$  for  $t \in [t^*, t_G]$ , then  $\mathcal{A}^G F(v) = F(v-) = F(t^*)$  for  $dG$ -a.e.  $v \in (t^*, t_G)$ . Thus,  $P$ -a.e. path of  $M$  is constant on  $[t^*, t_G]$ . It follows that  $M = M^{t^*}$   $P$ -a.s., and so by Lemma 3.1,  $M$  is a uniformly integrable  $\mathbb{F}^\gamma$ -martingale on  $[0, t_G]$ . If there is no such  $t^*$ , then there exists a strictly increasing sequence  $(t_n)_{n \in \mathbb{N}_0} \subset [0, t_G]$  such that

$$\lim_{n \rightarrow \infty} t_n = t_G \quad \text{and} \quad F(t_n) \neq F(t_{n-1}), \quad n \in \mathbb{N}.$$

For  $n \in \mathbb{N}$ , define the random time  $\tau_n : \Omega \rightarrow [0, t_G]$  by

$$\tau_n := t_n \mathbf{1}_{\{M_{t_n} - M_{t_{n-1}} \neq 0\}} + t_G \mathbf{1}_{\{M_{t_n} - M_{t_{n-1}} = 0\}}.$$

Since  $\{M_{t_n} - M_{t_{n-1}} \neq 0\} \in \mathcal{F}_{t_n}^M$ ,  $\tau_n$  is an  $\mathbb{F}^M$ -stopping time, and

$$M_{t_n} - M_{t_{n-1}} = \begin{cases} F(t_n) - F(t_{n-1}) \neq 0 & \text{if } \gamma > t_n, \\ \mathcal{A}^G F(\gamma) - F(t_{n-1}) & \text{if } t_{n-1} < \gamma \leq t_n, \\ 0 & \text{if } \gamma \leq t_{n-1}, \end{cases}$$

so that

$$\{\tau_n = t_G\} = \{M_{t_n} - M_{t_{n-1}} = 0\} \subset \{\gamma \leq t_n\} \subset \{M_{t_{n+1}} - M_{t_n} = 0\} = \{\tau_{n+1} = t_G\}.$$

This shows that the sequence  $(\tau_n)_{n \in \mathbb{N}}$  is increasing and satisfies

$$\lim_{n \rightarrow \infty} P[\tau_n = t_G] \geq \lim_{n \rightarrow \infty} P[\gamma \leq t_{n-1}] = P[\gamma < t_G] = 1; \quad (3.4)$$

here, we use the assumption  $\Delta G(t_G) = 0$ . Moreover, for  $n \in \mathbb{N}$  and  $s \in [0, t_G]$ , it follows from the definition of  $M$  that

$$\begin{aligned} M_s \mathbf{1}_{\{\gamma \leq t_n\}} &= F(s) \mathbf{1}_{\{s < \gamma\}} \mathbf{1}_{\{\gamma \leq t_n\}} + \mathcal{A}^G F(\gamma) \mathbf{1}_{\{s \geq \gamma\}} \mathbf{1}_{\{\gamma \leq t_n\}} \\ &= F(t_n \wedge s) \mathbf{1}_{\{t_n \wedge s < \gamma\}} \mathbf{1}_{\{\gamma \leq t_n\}} + \mathcal{A}^G F(\gamma) \mathbf{1}_{\{\gamma \leq t_n \wedge s\}} \mathbf{1}_{\{\gamma \leq t_n\}} \\ &= M_{t_n \wedge s} \mathbf{1}_{\{\gamma \leq t_n\}}. \end{aligned}$$

This together with  $\{\tau_n = t_G\} \subset \{\gamma \leq t_n\}$  gives

$$\begin{aligned} M_s^{\tau_n} &= M_s \mathbf{1}_{\{\tau_n = t_G\}} + M_{t_n \wedge s} \mathbf{1}_{\{\tau_n = t_n\}} = M_{t_n \wedge s} \mathbf{1}_{\{\gamma \leq t_n\}} \mathbf{1}_{\{\tau_n = t_G\}} + M_{t_n \wedge s} \mathbf{1}_{\{\tau_n = t_n\}} \\ &= M_{t_n \wedge s} \mathbf{1}_{\{\tau_n = t_G\}} + M_{t_n \wedge s} \mathbf{1}_{\{\tau_n = t_n\}} = M_s^{t_n}. \end{aligned}$$

Now the claim follows from (3.4) and the fact that by Lemma 3.1, each  $M^{t_n}$  is a uniformly integrable  $\mathbb{F}^\gamma$ -martingale on  $[0, t_G]$ .

(b) Lemmas 3.4 and 2.6 show that  $F$  is of infinite variation on  $[0, t_G]$ . Moreover, we have  $P[M_t = F(t), t \in [0, t_G]] \geq P[\gamma = t_G] = \Delta G(t_G) > 0$ . Now the claim follows from Lemma B.6.

(c) The assumption that  $M_{t_G}$  is integrable together with Lemma 3.4 gives  $\mathcal{A}^G F \in L^1(dG)$ . Since  $\Delta G(t_G) > 0$ , the limit  $F(t_G-) = \lim_{t \uparrow t_G} F(t)$  exists in  $\mathbb{R}$  by Lemma 2.6 and so there is  $C > 0$  such that  $\sup_{s \in [0, t_G]} |F(s)| \leq C$ . Thus,

$$\begin{aligned} E \left[ \sup_{0 \leq s < t_G} |M_s| \right] &= E \left[ \sup_{0 \leq s < t_G} (|F(s)| \mathbf{1}_{\{s < \gamma\}} + |\mathcal{A}^G F(\gamma)| \mathbf{1}_{\{s \geq \gamma\}}) \right] \\ &\leq C + \int_{(0, t_G]} |\mathcal{A}^G F(v)| dG(v) < \infty. \end{aligned} \quad (3.5)$$

Moreover, using the definition of  $\mathcal{A}^G F$  and  $\Delta G(t_G) > 0$  in the third equality,

$$\begin{aligned} \lim_{t \uparrow t_G} M_t &= \lim_{t \uparrow t_G} (F(t) \mathbf{1}_{\{t < \gamma\}} + \mathcal{A}^G F(\gamma) \mathbf{1}_{\{t \geq \gamma\}}) = F(t_G-) \mathbf{1}_{\{\gamma = t_G\}} + \mathcal{A}^G F(\gamma) \mathbf{1}_{\{\gamma < t_G\}} \\ &= \mathcal{A}^G F(\gamma) = M_{t_G} \quad P\text{-a.s.} \end{aligned}$$

Since  $M$  is an  $\mathbb{F}^\gamma$ -martingale on  $[0, t_G]$  by Lemma 3.1, combining (3.5) with the martingale convergence theorem shows that  $M$  is an  $\mathbb{F}^\gamma$ -martingale on the right-closed interval  $[0, t_G]$ .  $\square$

### 3.2 Sub- and supermartingale property on $[0, t_G]$

**Definition 3.6.** If  $M$  is integrable on  $[0, t_G]$ , the *change in mass of  $M$*  (on  $[0, t_G]$ ) is defined as

$$\Delta\mu := E[M_{t_G}] - E[M_0] = \int_{(0, t_G]} \mathcal{A}^G F(v) dG(v) - F(0).$$

If  $M$  is integrable on  $[0, t_G]$ , it is a strict local martingale whenever  $\Delta\mu \neq 0$ . The following result gives a formula that allows to compute  $\Delta\mu$  easily.

**Lemma 3.7.** *Suppose that  $M$  is integrable on  $[0, t_G]$ . Then*

$$\Delta\mu = - \lim_{t \uparrow t_G} F(t) \overline{G}(t) \mathbf{1}_{\{\Delta G(t_G)=0\}},$$

and for  $0 \leq s < t_G$ ,

$$E[M_{t_G} | \mathcal{F}_s^\gamma] - M_s = \frac{\Delta\mu}{\overline{G}(s)} \mathbf{1}_{\{s < \gamma\}} \quad P\text{-a.s.} \quad (3.6)$$

Moreover,  $M$  is always an integrable  $\mathbb{F}^M$ -local  $\mathbb{F}^\gamma$ -martingale on  $[0, t_G]$ , and more precisely,

- (a)  $M$  is an  $\mathbb{F}^\gamma$ -submartingale and a strict local martingale on  $[0, t_G]$  if and only if  $\Delta\mu > 0$ ,
- (b)  $M$  is an  $\mathbb{F}^\gamma$ -martingale on  $[0, t_G]$  if and only if  $\Delta\mu = 0$ ,
- (c)  $M$  is an  $\mathbb{F}^\gamma$ -supermartingale and a strict local martingale on  $[0, t_G]$  if and only if  $\Delta\mu < 0$ .

*Proof.* If  $\Delta G(t_G) > 0$ , then  $M$  is an  $H^1$ - $\mathbb{F}^\gamma$ -martingale on  $[0, t_G]$  by Theorem 3.5 (c), and all claims follow. So suppose that  $\Delta G(t_G) = 0$ . Then  $M$  is an  $\mathbb{F}^M$ -local  $\mathbb{F}^\gamma$ -martingale on  $[0, t_G]$  by Theorem 3.5 (a). Moreover, using  $\Delta G(t_G) = 0$  and (2.4) gives

$$\begin{aligned} \Delta\mu &= E[M_{t_G}] - E[M_0] = \int_{(0, t_G]} \mathcal{A}^G F(v) dG(v) - F(0) \\ &= \int_{(0, t_G]} \mathcal{A}^G F(v) dG(v) - F(0) = - \lim_{t \uparrow t_G} F(t) \overline{G}(t). \end{aligned}$$

To establish (3.6), fix  $0 \leq s < t_G$ . Using  $M_{t_G} = \mathcal{A}^G F(\gamma)$   $P$ -a.s., the fact that  $\{s < \gamma\}$  is an atom of  $\mathcal{F}_s^\gamma$ ,  $\Delta G(t_G) = 0$  and (2.4) gives

$$\begin{aligned} E[M_{t_G} | \mathcal{F}_s^\gamma] &= E[\mathcal{A}^G F(\gamma) | s < \gamma] \mathbf{1}_{\{s < \gamma\}} + \mathcal{A}^G F(\gamma) \mathbf{1}_{\{s \geq \gamma\}} \\ &= \frac{\int_{(s, t_G]} \mathcal{A}^G F(u) dG(u)}{1 - G(s)} \mathbf{1}_{\{s < \gamma\}} + \mathcal{A}^G F(\gamma) \mathbf{1}_{\{s \geq \gamma\}} \\ &= \frac{\int_{(s, t_G]} \mathcal{A}^G F(u) dG(u)}{1 - G(s)} \mathbf{1}_{\{s < \gamma\}} + \mathcal{A}^G F(\gamma) \mathbf{1}_{\{s \geq \gamma\}} \\ &= \frac{F(s) \overline{G}(s) - \lim_{t \uparrow t_G} F(t) \overline{G}(t)}{\overline{G}(s)} \mathbf{1}_{\{s < \gamma\}} + \mathcal{A}^G F(\gamma) \mathbf{1}_{\{s \geq \gamma\}} \\ &= \frac{F(s) \overline{G}(s) + \Delta\mu}{\overline{G}(s)} \mathbf{1}_{\{s < \gamma\}} + \mathcal{A}^G F(\gamma) \mathbf{1}_{\{s \geq \gamma\}} \\ &= M_s + \frac{\Delta\mu}{\overline{G}(s)} \mathbf{1}_{\{s < \gamma\}} \quad P\text{-a.s.} \end{aligned}$$

The remaining claims are straightforward.  $\square$

The next result shows that if  $M$  is integrable on  $[0, t_G]$ , then it can be naturally decomposed into its initial value  $M_0$  and the difference of two supermartingales starting at 0, i.e., it is a quasimartingale (cf. [8, Theorem VI.40]).

**Corollary 3.8.** *Let  $M = \mathcal{M}^G F$  be integrable on  $[0, t_G]$ . Set  $M^\uparrow := \mathcal{M}^G(F^\uparrow)$  and  $M^\downarrow := \mathcal{M}^G(F^\downarrow)$ . Then  $M^\uparrow$  and  $M^\downarrow$  are  $\mathbb{F}^\gamma$ -supermartingales on  $[0, t_G]$ , start at 0, and satisfy*

$$M = M_0 + M^\uparrow - M^\downarrow \quad P\text{-a.s.}$$

*Proof.* It follows from Lemmas 2.7 and 3.4 that  $M^\uparrow$  and  $M^\downarrow$  are well defined, integrable on  $[0, t_G]$  and start at 0. Nonnegativity of  $F^\uparrow, F^\downarrow$  and Lemma 3.7 give the supermartingale property. The decomposition result follows from the definitions of  $M, M^\uparrow$  and  $M^\downarrow$ , and from (2.5) and (2.6).  $\square$

### 3.3 $H^1$ -martingale property on $[0, t_G]$

If  $M$  is integrable on  $[0, t_G]$  and  $\Delta G(t_G) > 0$ , then  $M$  is automatically an  $H^1$ -martingale on  $[0, t_G]$  by Theorem 3.5. If  $\Delta G(t_G) = 0$ , however, the situation is more delicate.

**Lemma 3.9.** *Suppose that  $\Delta G(t_G) = 0$ . Then the following are equivalent:*

- (a)  *$M$  is an  $H^1$ -martingale on  $[0, t_G]$  (in the sense that there exists a filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty]}$  of  $\mathcal{F}$  such that  $M$  is an  $H^1$ - $\mathbb{F}$ -martingale on  $[0, t_G]$ , cf. Definition B.1).*
- (b)  *$M$  is an  $H^1$ - $\mathbb{F}^\gamma$ -martingale on  $[0, t_G]$ .*
- (c)  *$F(\cdot-) \in L^1(dG)$  and  $\mathcal{A}^G F \in L^1(dG)$ .*

*Proof.* “(b)  $\Rightarrow$  (a)”: This is trivial.

“(a)  $\Rightarrow$  (c)”: Lemma 3.4 yields  $\mathcal{A}^G F \in L^1(dG)$ . Moreover, using the definitions of  $\zeta^F$  and  $M$  in (3.2) and (3.3) and the fact that  $\Delta G(t_G) = 0$ , we obtain

$$\begin{aligned} \int_{(0, t_G)} |F(v-)| dG(v) &\leq \int_{(0, t_G)} \sup_{t < v} |F(t)| dG(v) = \int_{(0, t_G)} \sup_{t < v} |\zeta^F(t, v)| dG(v) \\ &\leq \int_{(0, t_G)} \sup_{t \in [0, t_G]} |\zeta^F(t, v)| dG(v) = E \left[ \sup_{t \in [0, t_G]} |M_t| \right] < \infty. \end{aligned}$$

“(c)  $\Rightarrow$  (b)”: As  $M$  is a local  $\mathbb{F}^\gamma$ -martingale on  $[0, t_G]$  by Theorem 3.5 (a), it suffices to show that  $E \left[ \sup_{t \in [0, t_G]} |M_t| \right] < \infty$ . First, note that  $|F(t)| \leq |F(0)| + |F|(t) \leq |F(0)| + |F|(v-)$  for  $0 \leq t < v < t_G$ . Thus, for  $t \in [0, t_G]$  and  $v \in (0, t_G)$ ,

$$|\zeta^F(t, v)| = |F(t)| \mathbf{1}_{\{t < v\}} + |\mathcal{A}^G F(v)| \mathbf{1}_{\{t \geq v\}} \leq |F(0)| + |F|(v-) + |\mathcal{A}^G F(v)|.$$

Using this together with the definition of  $M$  in (3.3) and the fact that  $\Delta G(t_G) = 0$ , we get

$$\begin{aligned} E \left[ \sup_{t \in [0, t_G]} |M_t| \right] &= \int_{(0, t_G)} \sup_{t \in [0, t_G]} |\zeta^F(t, v)| dG(v) \\ &\leq |F(0)| + \int_{(0, t_G)} |F|(v-) dG(v) + \int_{(0, t_G)} |\mathcal{A}^G F(v)| dG(v) < \infty. \quad \square \end{aligned}$$

As a corollary, we obtain a criterion which allows us to construct (uniformly integrable) martingales that are not  $H^1$ -martingales. A concrete example is given in Example 3.15 below.

**Corollary 3.10.** *Suppose that  $\Delta G(t_G) = 0$ . Assume that  $(\mathcal{A}^G F)^-$  or  $(\mathcal{A}^G F)^+$  belongs to  $L^1(dG)$  and that  $\lim_{t \uparrow t_G} F(t) \overline{G}(t) = 0$ , but that  $F(\cdot-) \notin L^1(dG)$ . Then  $M$  is an  $\mathbb{F}^\gamma$ -martingale but **not** an  $H^1$ -martingale on  $[0, t_G]$ .*

*Proof.* Lemmas 2.5, 3.4 and 3.7 show that  $M$  is an  $\mathbb{F}^\gamma$ -martingale on  $[0, t_G]$ . That  $M$  fails to be an  $H^1$ -martingale on  $[0, t_G]$  follows from Lemma 3.9.  $\square$

**Remark 3.11.** Even if  $M$  is an  $H^1$ -martingale on  $[0, t_G]$ , one may have  $F \notin L^1(dG)$ . Indeed, define  $G : \mathbb{R} \rightarrow [0, 1]$  by  $G(t) = \frac{1}{e-1} \sum_{k=1}^{\infty} \frac{1}{k!} \mathbf{1}_{[k, \infty)}(t)$  and  $F : [0, \infty) \rightarrow \mathbb{R}$  by  $F(t) = \sum_{k=1}^{\infty} (k-1)! \mathbf{1}_{[k, k+1)}(t)$ . Then  $t_G = \infty$  and for  $n \in \mathbb{N}$ ,

$$\begin{aligned} \int_{(0, \infty)} |F(v-)| dG(v) &= \sum_{n=1}^{\infty} F(n-1) \Delta G(n) = \frac{1}{e-1} \sum_{n=2}^{\infty} \frac{1}{n(n-1)} < \infty, \\ \int_{(0, \infty)} |F(v)| dG(v) &= \sum_{n=1}^{\infty} F(n) \Delta G(n) = \frac{1}{e-1} \sum_{n=1}^{\infty} \frac{1}{n} = \infty. \end{aligned}$$

So  $F(\cdot-) \in L^1(dG)$  but  $F \notin L^1(dG)$ . It remains to show that  $M$  is an  $H^1$ -martingale. In view of Lemma 3.9, the fact that  $F(\cdot-) \in L^1(dG)$  and the definition of  $\mathcal{A}^G F$ , this boils down to proving that

$$\int_{(0, \infty)} \left| \frac{dF}{dG}(v) \overline{G}(v) \right| dG(v) = \sum_{n=1}^{\infty} \left| \frac{\Delta F(n)}{\Delta G(n)} \right| \overline{G}(n) \Delta G(n) = \sum_{n=1}^{\infty} |\Delta F(n)| \overline{G}(n)$$

is finite. But this is true, because for  $n \in \mathbb{N}$ ,

$$\begin{aligned} |\Delta F(n)| &= F(n) - F(n-1) \leq F(n) = (n-1)!, \\ \overline{G}(n) &= 1 - G(n) = \frac{1}{e-1} \sum_{k=n+1}^{\infty} \frac{1}{k!} \leq \frac{e}{e-1} \frac{1}{(n+1)!}. \end{aligned}$$

### 3.4 Summary and examples

The flow chart in Figure 3.1 summarises the results of the previous sections. It gives the conditions one has to check in order to determine the (local) martingale properties of  $\mathcal{M}^G F$ . In this section, we give examples for four of the five cases one can end up with; examples for the fifth case that  $\mathcal{M}^G F$  is an  $H^1$ -martingale are easy to find (take, e.g.,  $F \stackrel{\text{loc}}{\ll} G$  bounded with  $\mathcal{A}^G F$  bounded).

**Example 3.12** (A process which fails to be a semimartingale). Let  $G : \mathbb{R} \rightarrow [0, 1]$  be given by  $G(t) = \frac{t}{2} \mathbf{1}_{[0, 1)}(t) + \mathbf{1}_{[1, \infty)}(t)$ , i.e., the law of the jump time  $\gamma$  is a mixture of a uniform distribution on  $(0, 1)$  and a Dirac measure at 1. In particular,  $t_G = 1$  and  $\Delta G(t_G) = \frac{1}{2}$ . The idea is to choose any  $F \stackrel{\text{loc}}{\ll} G$  that is of infinite variation on  $[0, t_G]$ . Then

$$P[M_t = F(t), t \in [0, t_G]] \geq P[\gamma = t_G] = \Delta G(t_G) = \frac{1}{2}$$

by the definition of  $\mathcal{M}^G F$ , and Lemma B.6 asserts that  $\mathcal{M}^G F$  fails to be a semimartingale on  $[0, t_G]$ . (Alternatively, one can use Lemma 2.6 to infer that  $\mathcal{A}^G F \notin L^1(dG)$  and then apply Lemma 3.4 and Theorem 3.5 (b).) A concrete example is given by  $F(t) = \mathbf{1}_{[0, 1)}(t) \sin \frac{1}{1-t}$ ,  $t \geq 0$ .

For the remaining examples, let  $G : \mathbb{R} \rightarrow [0, 1]$  be given by  $G(t) = t \mathbf{1}_{[0, 1)}(t) + \mathbf{1}_{[1, \infty)}(t)$ , i.e.,  $\gamma$  is uniformly distributed on  $(0, 1)$ . In particular,  $t_G = 1$  and  $\Delta G(t_G) = 0$ . Then for each  $F \stackrel{\text{loc}}{\ll} G$ ,  $\mathcal{M}^G F$  is a local martingale on  $[0, 1]$  by Theorem 3.5 (a).

**Example 3.13** (A strict local martingale that fails to be integrable). The idea is to find an  $F \stackrel{\text{loc}}{\ll} G$  such that  $\mathcal{M}_{t_G}^G F$  is not integrable or, equivalently by Lemma 3.4,  $\mathcal{A}^G F \notin L^1(dG)$ . A concrete example is given by  $F(t) = \mathbf{1}_{[0, 1)}(t) \sin \frac{1}{1-t}$ ,  $t \geq 0$ . Then  $F \stackrel{\text{loc}}{\ll} G$ ,

$$\mathcal{A}^G F(v) = \sin \frac{1}{1-v} - \frac{1}{1-v} \cos \frac{1}{1-v} \quad dG\text{-a.e. on } (0, 1),$$

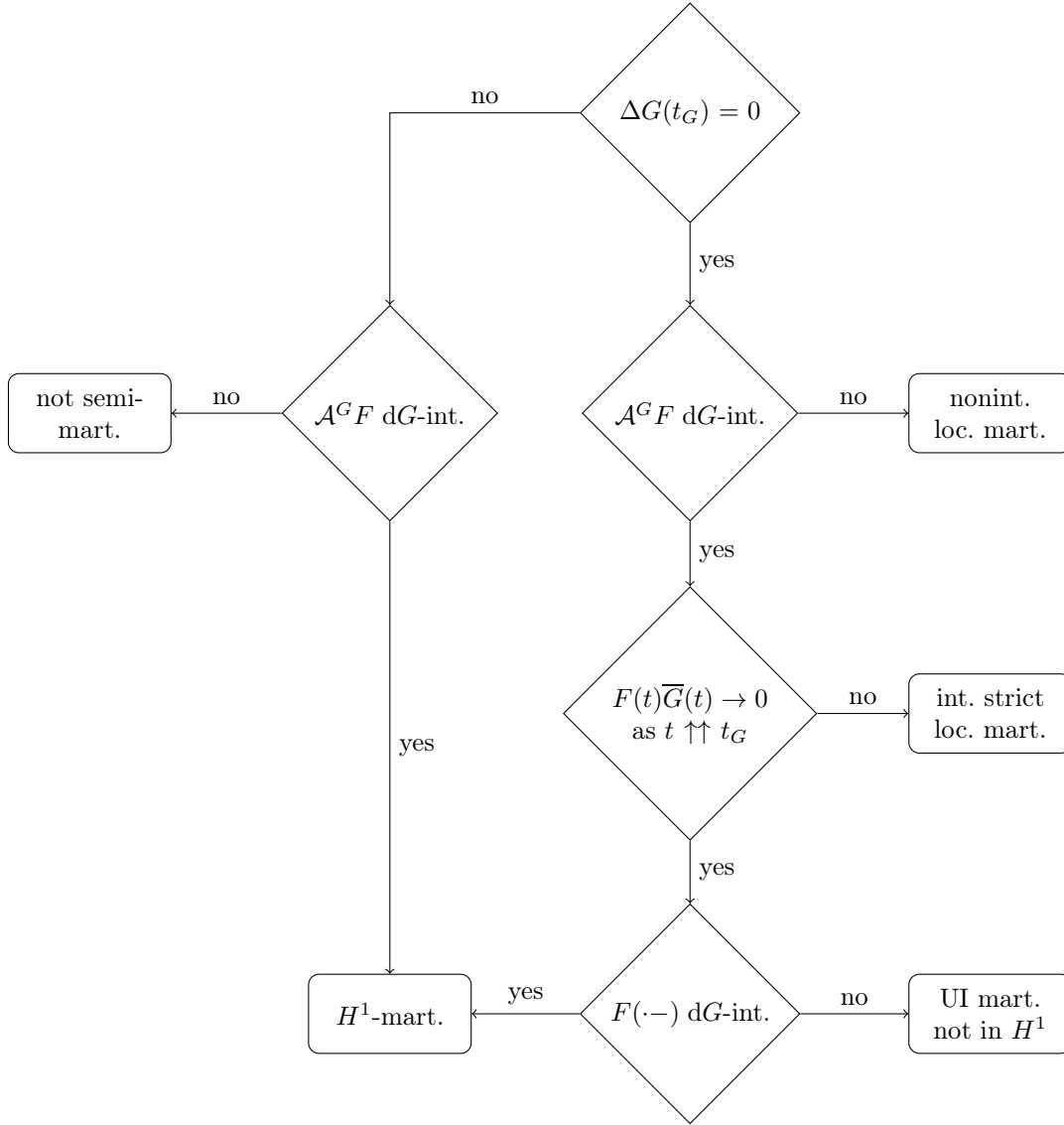


Figure 3.1: Decision diagram for single jump local martingales.

and one can show that  $\mathcal{A}^G F \notin L^1(dG)$ . Indeed, it suffices to show that  $\int_0^1 \left| \frac{1}{1-v} \cos \frac{1}{1-v} \right| dv = \infty$ , or equivalently,  $\int_1^\infty \left| \frac{\cos x}{x} \right| dx = \infty$ . But for each  $k \in \mathbb{N}$ ,

$$\int_{(2k-1)\frac{\pi}{2}}^{(2k+1)\frac{\pi}{2}} \left| \frac{\cos x}{x} \right| dx \geq \frac{1}{(2k+1)\frac{\pi}{2}} \int_{(2k-1)\frac{\pi}{2}}^{(2k+1)\frac{\pi}{2}} |\cos x| dx = \frac{2}{(2k+1)\frac{\pi}{2}}$$

and summing over  $k$  leads to an infinite series on the right-hand side.

**Example 3.14** (An integrable strict local martingale). Here the idea is to find  $F \ll^{\text{loc}} G$  with  $F(0) > 0$  and  $\mathcal{A}^G F = 0$  dG-a.e. on  $(0, 1)$ . This means that  $\mathcal{M}^G F$  starts at  $F(0) > 0$  at time 0 and ends up at zero at time 1  $P$ -a.s. Therefore,  $\mathcal{M}^G F$  cannot be a martingale on  $[0, 1]$ . A simple example is given by  $F(t) = \frac{1}{1-t} \mathbf{1}_{[0,1)}(t)$ ,  $t \geq 0$ . Then  $F \ll^{\text{loc}} G$ ,

$$\Delta\mu = -\lim_{t \uparrow \uparrow 1} F(t)\overline{G}(t) = -\lim_{t \uparrow \uparrow 1} \frac{1-t}{1-t} = -1 \quad \text{and} \quad \mathcal{A}^G F = 0 \quad \text{dG-a.e. on } (0, 1).$$

Thus, by Lemmas 3.4 and 3.7,  $\mathcal{M}^G F$  is an integrable strict local martingale and a supermartingale.

**Example 3.15** (A martingale that fails to be in  $H^1$ ). Finding  $F \ll^{\text{loc}} G$  such that  $\mathcal{M}^G F$  is a martingale on  $[0, 1]$  that fails to be in  $H^1$  is a bit tricky: if  $F$  grows too slowly, then  $\mathcal{M}^G F$  will be an  $H^1$ -martingale, but if  $F$  grows too quickly, then  $\mathcal{M}^G F$  will be a strict local martingale. The idea is to find an  $F \ll^{\text{loc}} G$  that satisfies the assumptions of Corollary 3.10. Define  $F : [0, \infty) \rightarrow \mathbb{R}$  by  $F(t) = \frac{1}{(1-t) \log \frac{e}{1-t}} \mathbf{1}_{[0,1)}(t)$ . Then  $F \ll^{\text{loc}} G$  is nonnegative and increasing on  $[0, 1)$ , and by monotone convergence,

$$\int_{(0,1)} |F(v-)| dG(v) = \lim_{t \uparrow 1} \int_0^t F(v) dv = \lim_{t \uparrow 1} \left[ \log \left( \log \frac{e}{1-v} \right) \right]_0^t = \lim_{t \uparrow 1} \log \left( \log \frac{e}{1-t} \right) = \infty.$$

Moreover,

$$\mathcal{A}^G F(v) = \frac{1}{(1-v) \left( \log \frac{e}{1-v} \right)^2} \text{ dG-a.e. on } (0, 1).$$

Thus,  $\mathcal{A}^G F$  is nonnegative, and therefore  $(\mathcal{A}^G F)^- \in L^1(dG)$ . Finally,

$$-\Delta\mu = \lim_{t \uparrow 1} F(t) \overline{G}(t) = \lim_{t \uparrow 1} \frac{1}{\log \frac{e}{1-t}} = 0.$$

Hence, by Corollary 3.10,  $\mathcal{M}^G F$  is a (uniformly integrable) martingale on  $[0, 1]$  but not in  $H^1$ .

## 4 A counter-example in stochastic integration

In this section, we consider the following problem from stochastic integration: Does there exist a pair  $(M, H)$ , where  $M = (M_t)_{t \in [0,1]}$  is a (true) martingale and  $H = (H_t)_{t \in [0,1]}$  an integrand with  $0 \leq H \leq 1$  such that the stochastic integral  $H \bullet M$  is a *strict* local martingale? By the BDG inequality,  $H \bullet M$  is again a martingale if  $H$  is bounded and  $M$  is an  $H^1$ -martingale. Nevertheless, the answer to the above question is positive as is shown in [17, Corollaire VI.21] by an abstract existence proof using the Baire category theorem. It took, however, 30 years until a quite ingenious concrete example was published by Cherny [2]. He constructed the martingale integrator  $M = (M_n)_{n \in \mathbb{N}}$  recursively as follows. Starting with  $M_0 = 1$ ,  $M$  moves up or down at time 1. If it moves down, it stays constant afterwards and if it moves up,  $M$  can again move up or down at time 2, and so on. The precise magnitudes of up and down movements are chosen such that  $M$  becomes a uniformly integrable martingale that is not in  $H^1$ . Note that the structure of  $M$  is precisely of the form  $\mathcal{M}^G F$ . Indeed, if  $\gamma$  denotes the time when  $M$  moves down and  $G$  is the distribution function of  $\gamma$ , we can find a function  $F \ll^{\text{loc}} G$  such that the piecewise constant extension of  $M$  to  $[0, \infty)$  equals  $\mathcal{M}^G F$ . The integrand in Cherny's example is, up to a time transformation, the same as we use in Theorem 4.2 below. The goal of this section is to provide an example of this kind, which works for  $G$  the uniform distribution and any nondecreasing function  $F$  such that  $\mathcal{M}^G F$  is a uniformly integrable martingale but not in  $H^1$ .

In preparation of our counter-example, we first show that the class of single jump local martingales is closed under stochastic integration with bounded deterministic integrands.

**Proposition 4.1.** *Let  $F \ll^{\text{loc}} G$  be such that  $\mathcal{M}^G F$  is a local martingale on  $[0, t_G]$ . Moreover, let  $J : [0, \infty) \rightarrow \mathbb{R}$  be a bounded Borel-measurable function. Define  $F^J : [0, \infty) \rightarrow \mathbb{R}$  by  $F^J(t) = \int_{(0,t]} J(u) dF(u)$  for  $t \in [0, t_G)$  and  $F^J(t) = 0$  for  $t \geq t_G$ . Then  $F^J \ll^{\text{loc}} G$  and*

$$J \bullet \mathcal{M}^G F = \mathcal{M}^G F^J \text{ P-a.s.}$$



*Proof.* We only establish the result for the case  $\Delta G(t_G) = 0$ , which corresponds to the setting of Theorem 4.2 below. Since  $J$  is bounded and  $F \ll_{\text{loc}}^{\text{loc}} G$ ,  $J \in L^1_{\text{loc}}(\text{d}F)$ , so that  $F^J$  is well defined. Clearly,  $F^J \ll_{\text{loc}}^{\text{loc}} G$  with local density  $\frac{\text{d}F^J}{\text{d}G} = J \frac{\text{d}F}{\text{d}G}$ . Fix  $t \in [0, t_G]$ . Using the definition of  $\mathcal{M}^G F$ , on  $\{t < \gamma\}$ ,

$$J \bullet \mathcal{M}^G F_t = \int_{(0,t]} J(u) \text{d}F(u) = F^J(t) = \mathcal{M}_t^G F^J \quad P\text{-a.s.} \quad (4.1)$$

Using dominated convergence, on  $\{t \geq \gamma\} \cap \{\gamma < t_G\}$ ,

$$\begin{aligned} J \bullet \mathcal{M}^G F_t &= \int_{(0,\gamma)} J(u) \text{d}\mathcal{M}^G F_u + J(\gamma) \Delta \mathcal{M}^G F_\gamma \\ &= \int_{(0,\gamma)} J(u) \text{d}F(u) + J(\gamma) (\mathcal{A}^G F(\gamma) - F(\gamma-)) \\ &= F^J(\gamma-) - J(\gamma) \frac{\text{d}F}{\text{d}G}(\gamma) \overline{G}(\gamma) = F^J(\gamma-) - \frac{\text{d}F^J}{\text{d}G}(\gamma) \overline{G}(\gamma) \\ &= \mathcal{A}^G F^J(\gamma) = \mathcal{M}_t^G F^J \quad P\text{-a.s.} \end{aligned} \quad (4.2)$$

As  $\Delta G(t_G) = 0$ ,  $\gamma < t_G$   $P$ -a.s. and (4.1) together with (4.2) completes the proof.  $\square$

Throughout the rest of this section, let  $G : \mathbb{R} \rightarrow [0, 1]$  be given by  $G(t) = t \mathbf{1}_{[0,1)}(t) + \mathbf{1}_{[1,\infty)}(t)$ , i.e.,  $\gamma$  is uniformly distributed on  $(0, 1)$ . In particular,  $t_G = 1$  and  $\Delta G(t_G) = 0$ . Moreover, we always consider the filtration  $\mathbb{F}^\gamma$  introduced in Section 3.

**Theorem 4.2.** *Let  $F \ll_{\text{loc}}^{\text{loc}} G$  be such that  $\mathcal{M}^G F$  is a martingale on  $[0, 1]$  which is not in  $H^1$  (cf. Corollary 3.10). Suppose that  $F$  is nondecreasing on  $(0, 1)$  and satisfies  $F(0) = 0$ . Define the deterministic integrand  $J = (J_t)_{t \in [0,1]}$  by*

$$J = \sum_{n=0}^{\infty} \mathbf{1}_{\llbracket 1-2^{-2n}, 1-2^{-(2n+1)} \rrbracket}.$$

*Then the stochastic integral  $J \bullet \mathcal{M}^G F$  is a strict local martingale on  $[0, 1]$ .*

A concrete example for a function  $F$  satisfying all conditions of Theorem 4.2 is given in Example 3.15. The choice of  $J$  is inspired by Cherny [2].

*Proof.* Proposition 4.1 yields  $F^J \ll_{\text{loc}}^{\text{loc}} G$  and  $J \bullet \mathcal{M}^G F = \mathcal{M}^G F^J$   $P$ -a.s. By Theorem 3.5,  $\mathcal{M}^G F^J$  is a local  $\mathbb{F}^\gamma$ -martingale on  $[0, 1]$ . It will be a strict local martingale if  $\mathcal{M}_1^G F^J$  is not integrable, which holds by Lemma 3.4 if and only if

$$\int_{(0,1)} |\mathcal{A}^G F^J(t)| \text{d}G(t) = \int_0^1 \left| F^J(t) - J(t) \frac{\text{d}F}{\text{d}G}(t)(1 - G(t)) \right| \text{d}t = \infty. \quad (4.3)$$

Since  $\mathcal{M}^G F$  is a martingale on  $[0, 1]$ ,  $\mathcal{A}^G F \in L^1(\text{d}G)$  by Lemma 3.4 and so

$$\begin{aligned} \int_0^1 \left| J(t)F(t) - J(t) \frac{\text{d}F}{\text{d}G}(t)(1 - G(t)) \right| \text{d}t &= \int_0^1 J(t) |\mathcal{A}^G F(t)| \text{d}t \\ &\leq \int_0^1 |\mathcal{A}^G F(t)| \text{d}t = \int_{(0,1)} |\mathcal{A}^G F(t)| \text{d}G(t) < \infty. \end{aligned}$$

In order to establish (4.3), it thus suffices to show that

$$\int_0^1 |F^J(t) - J(t)F(t)| \text{d}t = \int_0^1 ((1 - J(t))F^J(t) + J(t)F^{1-J}(t)) \text{d}t = \infty. \quad (4.4)$$

For  $n \in \mathbb{N}_0$ , set  $t_n := 1 - 2^{-n}$  and  $t_{-1} := -1$ . We note that  $t_{m+1} - t_m = \frac{1}{2}(t_m - t_{m-1}) = \frac{1}{3}(t_{m+1} - t_{m-1})$  for  $m \in \mathbb{N}_0$  and that  $F^J$  and  $F^{1-J}$  are constant on  $\{J = 0\}$  and  $\{J = 1\}$ , respectively. Using this, the nonnegativity of  $F^{1-J}$  and  $F$ , the fact that  $F = F^J + F^{1-J}$  is nondecreasing on  $(0, 1)$ , we obtain

$$\begin{aligned} & \int_0^1 \left( (1 - J(t))F^J(t) + J(t)F^{1-J}(t) \right) dt \\ &= \sum_{n=0}^{\infty} \left( F^J(t_{2n+1})(t_{2n+2} - t_{2n+1}) + F^{1-J}(t_{2n+1})(t_{2n+1} - t_{2n}) \right) \\ &\geq \frac{1}{2} \sum_{n=0}^{\infty} F(t_{2n+1})(t_{2n+1} - t_{2n}) = \frac{1}{6} \sum_{n=0}^{\infty} F(t_{2n+1})(t_{2n+1} - t_{2n-1}) \geq \frac{1}{6} \int_{(0,1)} F(t) dG(t). \end{aligned} \quad (4.5)$$

Since  $\mathcal{M}^G F$  is not in  $H^1$  by assumption, but  $\mathcal{M}^G F \in L^1(dG)$ , we get  $F(\cdot) \notin L^1(dG)$  by Lemma 3.9. This implies  $F \notin L^1(dG)$  because  $F$  is nondecreasing. Combining this with (4.5) shows that (4.4) holds true.  $\square$

## A Elements of real analysis

**Definition A.1.** Let  $T \in (0, \infty]$ . A function  $L : [0, \infty) \rightarrow \mathbb{R}$  is called a *left-continuous step function* on  $[0, T)$  if it is of the form

$$L = \sum_{j=1}^k a_j \mathbf{1}_{(t_{j-1}, t_j]},$$

where  $k \in \mathbb{N}$ ,  $0 \leq t_0 < t_1 < \dots < t_k < T$  and  $a_j \in \mathbb{R}$ ,  $j = 1, \dots, k$ . If  $F : [0, T] \rightarrow \mathbb{R}$  is any other function, we define the *elementary integral of  $L$  with respect to  $F$  on  $(0, T]$*  by

$$\int_{(0,T]} L(t) dF(t) := \sum_{j=1}^k a_j (F(t_j) - F(t_{j-1})).$$

The following result is an easy exercise in analysis.

**Lemma A.2.** Let  $T \in (0, \infty]$  and  $F : [0, T] \rightarrow \mathbb{R}$  be a function which is of infinite variation on  $[0, T]$ . Then for each  $n \in \mathbb{N}$ , there exists a left-continuous step function  $L_n : [0, \infty) \rightarrow \mathbb{R}$  on  $[0, T)$  with  $\sup_{t \geq 0} |L_n(t)| \leq 1/n$  and

$$\int_{(0,T]} L_n(t) dF(t) \geq 1.$$

## B Elements of (semi-)martingale theory

Throughout this section, we fix a probability space  $(\Omega, \mathcal{A}, P)$  and a time horizon  $T^* \in (0, \infty]$ . Moreover,  $I \subset [0, T^*]$  is assumed to be an interval of the form  $[0, T)$  or  $[0, T]$  for some  $T \in (0, T^*]$ .

**Definition B.1.** Fix a stochastic process  $X = (X_t)_{t \in [0, T^*]}$  and a filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty]}$  of  $\mathcal{A}$ .

- (a)  $X$  is of *finite variation* on  $I$  if for  $P$ -a.e.  $\omega$ , the function  $t \mapsto X_t(\omega)$  is of finite variation and càdlàg on  $I$ .
- (b)  $X$  is called  *$\mathbb{F}$ -adapted* on  $I$  if  $X_t$  is  $\mathcal{F}_t$ -measurable for each  $t \in I$ .
- (c)  $X$  is called *integrable* on  $I$  if  $E[|X_t|] < \infty$  for each  $t \in I$ .

- (d)  $X$  is an  $\mathbb{F}$ -(sub/super)martingale on  $I$  if for  $P$ -a.e.  $\omega$ , the function  $t \mapsto X_t(\omega)$  is càdlàg on  $I$  and  $X$  is  $\mathbb{F}$ -adapted on  $I$ , integrable on  $I$ , and satisfies the  $\mathbb{F}$ -(sub/super)martingale property on  $I$ , i.e.,

$$E[X_t | \mathcal{F}_s] (\geq, \leq) = X_s \quad \text{for all } s \leq t \text{ in } I.$$

$X$  is a (sub/super)martingale on  $I$  if there exists a filtration  $\mathbb{F}' = (\mathcal{F}'_t)_{t \in [0, \infty]}$  of  $\mathcal{A}$  such that  $X$  is an  $\mathbb{F}'$ -(sub/super)martingale on  $I$ .

- (e)  $X$  is an  $H^1$ - $\mathbb{F}$ -martingale on  $I$  if  $X$  is an  $\mathbb{F}$ -martingale on  $I$  and

$$E \left[ \sup_{t \in I} |X_t| \right] < \infty.$$

$X$  is an  $H^1$ -martingale if there exists a filtration  $\mathbb{F}' = (\mathcal{F}'_t)_{t \in [0, \infty]}$  of  $\mathcal{A}$  such that  $X$  is an  $H^1$ - $\mathbb{F}'$ -martingale on  $I$ .

- (f) Let  $\mathbb{G} = (\mathcal{G}_t)_{t \in [0, \infty]}$  be a subfiltration of  $\mathbb{F}$ .  $X$  is a  $\mathbb{G}$ -local  $\mathbb{F}$ -martingale on  $I$  if there exists an increasing sequence of  $\mathbb{G}$ -stopping times  $(\tau_n)_{n \in \mathbb{N}}$  with values in  $I \cup \{T\}$  such that for each  $n \in \mathbb{N}$ ,  $X^{\tau_n}$  is an  $\mathbb{F}$ -martingale on  $I$ , and
- (i) in case of  $T \notin I$ ,  $\lim_{n \rightarrow \infty} \tau_n = T$   $P$ -a.s.,
  - (ii) in case of  $T \in I$ ,  $\lim_{n \rightarrow \infty} P[\tau_n = T] = 1$ .

In both cases, the sequence  $(\tau_n)_{n \in \mathbb{N}}$  is called a  $\mathbb{G}$ -localising sequence (for  $X$ ). An  $\mathbb{F}$ -local  $\mathbb{F}$ -martingale on  $I$  is simply called a *local  $\mathbb{F}$ -martingale on  $I$* .  $X$  is a *local martingale on  $I$*  if there exists a filtration  $\mathbb{F}' = (\mathcal{F}'_t)_{t \in [0, \infty]}$  of  $\mathcal{A}$  such that  $X$  is a local  $\mathbb{F}'$ -martingale on  $I$ .  $X$  is a *strict local martingale on  $I$*  if it is a local martingale on  $I$ , but not a martingale on  $I$ .

- (g)  $X$  is an  $\mathbb{F}$ -semimartingale on  $I$  if there are processes  $M = (M_t)_{t \in I}$  and  $A = (A_t)_{t \in I}$  such that  $X = M + A$ , where  $M$  is local  $\mathbb{F}$ -martingale on  $I$  and  $A$  is  $\mathbb{F}$ -adapted on  $I$  and of finite variation on  $I$ .  $X$  is a *semimartingale on  $I$*  if there exists a filtration  $\mathbb{F}' = (\mathcal{F}'_t)_{t \in [0, \infty]}$  of  $\mathcal{A}$  such that  $X$  is an  $\mathbb{F}'$ -semimartingale on  $I$ .

Whenever we drop the qualifier “on  $I$ ” in the above notations it is understood that  $I = [0, T^*]$ .

The following result is a standard exercise in probability theory.

**Proposition B.2.** Fix a stochastic process  $X = (X_t)_{t \in [0, \infty]}$  and a filtration  $(\mathcal{F}_t)_{t \in [0, \infty]}$  of  $\mathcal{A}$ .

- (a) Let  $\mathbb{F}' = (\mathcal{F}'_t)_{t \in [0, \infty]}$  be a subfiltration of  $\mathbb{F}$  such that  $X$  is  $\mathbb{F}'$ -adapted on  $I$ . If  $X$  is an  $\mathbb{F}$ -(sub/super)martingale on  $I$ , then  $X$  is also an  $\mathbb{F}'$ -(sub/super)martingale on  $I$ .
- (b) Let  $\mathbb{G} = (\mathcal{G}_t)_{t \in [0, \infty]}$ ,  $\mathbb{G}' = (\mathcal{G}'_t)_{t \in [0, \infty]}$ ,  $\mathbb{F}' = (\mathcal{F}'_t)_{t \in [0, \infty]}$  be subfiltrations of  $\mathbb{F}$  satisfying  $\mathcal{G}_t \subset \mathcal{G}'_t \subset \mathcal{F}'_t \subset \mathcal{F}_t$  for each  $t \in I$  and such that  $X$  is  $\mathbb{F}'$ -adapted on  $I$ . If  $X$  is a  $\mathbb{G}$ -local  $\mathbb{F}$ -martingale on  $I$ , then it is also a  $\mathbb{G}'$ -local  $\mathbb{F}'$ -martingale on  $I$ .

**Definition B.3.** Let  $T \in (0, \infty]$  and  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty]}$  be a filtration of  $\mathcal{A}$ . A stochastic process  $L = (L_t)_{t \geq 0}$  is called an  $\mathbb{F}$ -elementary process on  $[0, T)$  if there exist  $\mathbb{F}$ -stopping times  $\tau_0, \dots, \tau_n$  with  $0 \leq \tau_0 \leq \tau_1 \leq \dots \leq \tau_n < T$  and bounded random variables  $A_1, \dots, A_n$  with each  $A_j$  being  $\mathcal{F}_{\tau_{j-1}}$ -measurable such that

$$L = \sum_{j=1}^k A_j \mathbf{1}_{[\tau_{j-1}, \tau_j]}.$$

Note that for each  $\omega \in \Omega$ , the path  $L_\cdot(\omega)$  is a left-continuous step function on  $[0, T)$ .

**Definition B.4.** Let  $T \in (0, \infty)$ . Let  $X = (X_t)_{t \in [0, \infty]}$  be a stochastic process,  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty]}$  a filtration of  $\mathcal{A}$  with respect to which  $X$  is adapted on  $[0, T]$  and  $L = (L_t)_{t \geq 0}$  an  $\mathbb{F}$ -elementary process on  $[0, T]$ . Define the  $\mathbb{F}$ -elementary stochastic integral of  $L$  with respect to  $X$  on  $(0, T]$  by

$$\left( \int_{(0, T]} L_t dX_t \right) (\omega) := \int_{(0, T]} L_t(\omega) dX_t(\omega).$$

**Lemma B.5.** Let  $T \in (0, \infty)$ . Let  $X = (X_t)_{t \in [0, \infty]}$  be a stochastic process and  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty]}$  a filtration of  $\mathcal{A}$ . If  $X$  is an  $\mathbb{F}$ -semimartingale on  $[0, T]$ , then for all  $\mathbb{F}$ -elementary processes  $(L_t^n)_{t \in [0, \infty]}$  on  $[0, T]$  satisfying  $\lim_{n \rightarrow \infty} \sup_{t \geq 0, \omega \in \Omega} |L_t^n(\omega)| = 0$ ,

$$P\text{-}\lim_{n \rightarrow \infty} \int_{(0, T]} L_t^n dX_t = 0.$$

*Proof.* This follows immediately from [19, Proposition 7.1.7] which is stronger than our result. Note that [19] work with *general* filtrations which need not satisfy the usual conditions.  $\square$

**Lemma B.6.** Let  $X = (X_t)_{t \in [0, \infty]}$  be a right-continuous stochastic process and  $T \in (0, \infty)$ . Suppose that there exists a deterministic function  $F : [0, T] \rightarrow \mathbb{R}$  such that  $F$  is of infinite variation on  $[0, T]$  and  $P[X(t) = F(t), t \in [0, T]] =: \epsilon > 0$ . Then  $X$  is **not** a semimartingale on  $[0, T]$ .

*Proof.* Seeking a contradiction, suppose there exists a filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty]}$  of  $\mathcal{A}$  with respect to which  $X$  is a semimartingale on  $[0, T]$ . Since  $F$  is of infinite variation on  $[0, T]$ , by Lemma A.2, for each  $n \in \mathbb{N}$  there exists a left-continuous step function  $L_n : [0, \infty) \rightarrow \mathbb{R}$  on  $[0, T]$  with  $\sup_{t \geq 0} |L_n(t)| \leq 1/n$  and

$$\int_{(0, T]} L_n(t) dF(t) \geq 1.$$

For each  $n \in \mathbb{N}$ , define the  $\mathbb{F}$ -elementary process  $(L_t^n)_{t \in [0, \infty)}$  on  $[0, T]$  by  $L_t^n(\omega) := L_n(t)$ . Then  $\lim_{n \rightarrow \infty} \sup_{t \geq 0, \omega \in \Omega} |L_t^n(\omega)| = 0$ , but

$$\begin{aligned} P \left[ \int_{(0, T]} L_t^n dX_t \geq 1 \right] &\geq P \left[ \int_{(0, T]} L_n(t) dF(t) \geq 1, X(t) = F(t), t \in [0, T] \right] \\ &= P[X(t) = F(t), t \in [0, T]] = \epsilon \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

This implies in particular that  $(\int_{(0, T]} L_n(t) dX_t)_{n \in \mathbb{N}}$  does not converge to 0 in probability. Hence,  $X$  fails to be an  $\mathbb{F}$ -semimartingale on  $[0, T]$  by Lemma B.5, and we arrive at a contradiction.  $\square$

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